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# The Effect of Pressure Gradient Force on an Accretion Disk Surrounding a Black-Hole

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A stationary thin-orbiting accretion disk surrounding a black-hole has been studied taking the pressure gradient force into account. We have found that under the condition appropriate for a binary $X$-ray source, with usual assumptions about viscous stress, the disk is largely modified when the pressure gradient force is taken into account. Therefore, current thin-disk models to account for observations of Cyg $X$-1 are self-inconsistent, since the pressure gradient force is neglected in those accretion disk models.

## § 1. Introduction

A number of accretion disk models[1),2)] have been proposed to account for observational properties of Cyg $X$-1, such as the emission of a power law spectrum of $X$-rays extending to hard $X$-ray range, time variations in $X$-ray intensity particularly as short as a few milliseconds and the existence of bimodal states observed as a major decrease (and following increase) of $X$-ray emission in $2\sim6$ KeV range. The structures of those accretion disks have all been calculated on the basis of the "standard accretion disk model" (hereafter referred to as SADM). One of the main assumptions in SADM is that azimuthal motion of accretion disk matter is replaced by Keplerian motion, that is, the pressure gradient force is ignored in the equation of motion. In the solution obtained from SADM we see that the neglect of the pressure gradient force is valid only if the geometric thickness of the accretion disk is sufficiently thin.

Recently, one of the authors (N.S.) has prepared the numerical code to solve detailed structures of accretion disks. We have found that preliminary numerical calculations do not reproduce the result derived from SADM even for a sufficiently thin accretion disk. Both numerical calculation and our analysis concerning an accretion disk dominated by gas pressure indicate that the pressure gradient force in the equation of motion couples to the viscous stress based on the $\alpha$-model and alters the structure of the disk completely. In order to show that this new result is not due to an artifact of our numerical code, we will discuss in § 4, by means of an analytic manner, the reason why SADM becomes invalid, for the particular example of an optically thick accretion disk dominated by gas pressure.

Most of equations used in the present paper have been derived in our previous paper.[3)] Basic equations and necessary physical data will be prepared in § 2. The physical parameters of the disk based on SADM will be briefly discussed in § 3. In § 4 the structure of the disk with the pressure gradient force will be discussed for an optically thick and gas pressure dominant case.

## § 2. Basic equations

To describe axisymmetric disk configuration of an accretion disk, we employ a cylindrical system of coordinate $(r, \varphi, z)$ with the $z$-axis chosen as the axis of rotation. Let the mass of the central black-hole and the accretion rate be $M$ and $\dot{M}$. Conservation laws of mass, momentum and energy are

$$2\pi r v_r S = \dot{M}\,, \tag{1}$$

$$v_r \frac{dv_r}{dr} - \frac{v_\varphi^2}{r} = -\frac{GM}{r^2} - \frac{1}{S}\frac{dW}{dr}\,, \tag{2}$$

$$S v_r \left\{\frac{dv_\varphi}{dr} + \frac{v_\varphi}{r}\right\} = -\frac{1}{r^2}\frac{d}{dr}(r^2 W_{r\varphi})\,, \tag{3}$$

$$\frac{d}{dr}\left\{\dot{M}\left(\frac{\gamma}{\gamma-1}\frac{W}{S} + \frac{1}{2}v_r^2 + \frac{1}{2}v_\varphi^2 - \frac{GM}{r}\right) + 2\pi r^2 W_{r\varphi}\Omega\right\} - 2\pi r E_R = 0\,, \tag{4}$$

where $S$, $W$, $W_{r\varphi}$ and $E_R$ are the density, pressure, viscous stress and energy loss rate integrated over the $z$-coordinate, respectively. For example, $S$ is given by $2\int_0^{z_0} \rho\, dz$, where $z_0$ is the thickness of the disk from the equatorial plane to the surface. The inward radial velocity and the azimuthal one are denoted by $v_r$ and $v_\varphi$, $\Omega$ the angular velocity and $\gamma$ the ratio of specific heats. In addition to the above equations, we have an equation which describes hydrostatic balance of disk matter along the $z$-coordinate. In terms of a polytropic assumption which connects pressure and density along the $z$-coordinate, vertical distribution of disk matter is solved analytically, from which one can calculate vertically integrated quantities, $S$, $W$, $W_{r\varphi}$ and $E_R$. Another important relation derived is[3]

$$\frac{W}{S} = \frac{I(N+1)}{2I(N)(N+1)}\frac{GM}{r}\left(\frac{z_0}{r}\right)^2 = A\frac{GM}{r}\left(\frac{z_0}{r}\right)^2\,, \tag{5}$$

where

$$I(N) = (2^N N!)^2/(2N+1)!\,, \tag{6}$$

and $N$ is the polytropic index defined by $P = K\rho^{1+1/N}$.

Since we are of special interest in an optically thick disk dominated by gas pressure, the integrated viscous stress and energy loss rate are writtens as[3]

$$W_{r\varphi} = \frac{2\alpha}{3}\left(\frac{GM}{r}\right)^{-1/2} W r^2 \frac{d\Omega}{dr}\,, \tag{7}$$

$$E_R = \frac{ac}{3\kappa}\frac{I(N+1)}{(N+1)^5}\left(\frac{m_H}{2k}\right)^4\left(\frac{GM}{r}\right)^5\left(\frac{z_0}{r}\right)^{10}\frac{1}{W}, \tag{8}$$

where $m_H$ is the proton mass, $\kappa$ the opacity which is, hereafter, assumed to be constant and $\alpha$ is so-called $\alpha$-parameter believed to lie between $10^{-3}$ to 1. Recently, Ichimaru[2),4)] has developed a theory of hydrodynamic turbulence and has derived an explicit formulation of viscous stress. In the case where the gas pressure prevails in the accretion disk, Ichimaru's formula reduces to Eq. (7) and $\alpha$ is determined to be $\sim 2/3$[3)] though it depends weakly on the polytropic index $N$.

## § 3. Standard accretion disk model

Before discussing the effect of the pressure gradient force on the structure of the accretion disk, let us briefly consider the disk structure determined from SADM. In SADM the first term and the pressure gradient term (last term) are ignored in Eq. (2), that is, Keplerian motion is assumed for azimuthal velocity. Furthermore, SADM neglects the first two terms in Eq. (4). It can be shown that these approximations are justified if the disk in question is geometrically thin, $z_0/r \ll 1$. Under the SADM approximation basic equations, Eqs. (1)$\sim$(4), are reduced to a set of algebraic equations, since Eq. (3) can easily be integrated (integration constant is put to zero). The solution to these algebraic equations is

$$W_{\mathrm{SD}} = \frac{1}{2\pi\alpha}\dot{M}\left(\frac{GM}{r}\right)^{1/2} r^{-1}. \tag{9}$$

$$v_{r,\mathrm{SD}} = \frac{\alpha I(N+1)}{2(N+1)I(N)}\left(\frac{z_0}{r}\right)^2\left(\frac{GM}{r}\right)^{1/2}, \tag{10}$$

$$\left(\frac{z_0}{r}\right)_{\mathrm{SD}} = \left\{\frac{18}{\alpha\pi^2}\frac{\kappa}{ac}\frac{(N+1)^5}{I(N+1)}\left(\frac{k}{m_H}\right)^4 \dot{M}^2 (GM)^{-7/2} r^{1/2}\right\}^{1/10}, \tag{11}$$

where suffix *SD* refers to SADM. In terms of Eqs. (9) and (11), one can evaluate the correction to $v_\varphi$ ignored in SADM,

$$v_\varphi = \sqrt{\frac{GM}{r}}\left\{1 - \frac{3A}{2}\left(\frac{z_0}{r}\right)^2 + 0\left\{\left(\frac{z_0}{r}\right)^4\right\}\right\}^{1/2}. \tag{12}$$

Clearly from Eq. (12), SADM approximation seems to be justified, if $z_0/r \ll 1$ is fulfilled throughout all domains of the accretion disk. However, as will be discussed in the next section, even in an accretion disk fully satisfying the condition $z_0/r \ll 1$, the azimuthal velocity $v_\varphi$ as well as the pressure $W$ deviates largely from Keplerian motion and $W_{\mathrm{SD}}$, respectively.

## § 4. Effect of the pressure gradient force

In order to present calculations in a more succinct way, we introduce non-dimensional variables,

$$v_\varphi=\sqrt{\frac{GM}{r}}f\,,\quad W=W_{\mathrm{SD}}g^{-1}\,. \tag{13}$$

which describe the deviations from SADM. In the following, basic equations are expressed in terms of $f$, $g$ and $y(\equiv z_0/r)$. With the aid of Eqs. (1), (5) and (13), the radial velocity can be written as

$$v_r=\alpha A\left(\frac{GM}{r}\right)^{1/2}y^2g\,, \tag{14}$$

where $A$ has been defined in Eq. (5), and the basic equations can be written as

$$\frac{d\ln g}{d\ln r}=-\frac{1}{Ay^2}\left\{\frac{f^2-1+2(\alpha A)^2y^4g^2(1-(d\ln y/d\ln r))}{1-\alpha^2Ay^2g^2}\right\}-\frac{3}{2}\,, \tag{15}$$

$$\frac{d\ln f}{d\ln r}=\frac{3}{2}(1-g)\,. \tag{16}$$

In deriving Eq. (16), Eq. (3) has been integrated and the integration constant has been put to zero (as for the details see our previous paper[3)]). The conservation law of energy is written as

$$\frac{d}{dr}\left\{\left(\frac{\gamma}{\gamma-1}Ay^2+\frac{(\alpha A)^2}{2}y^4g^2-\frac{f^2}{2}-1\right)\frac{GM}{r}\right\}-\frac{3}{2}y_{\mathrm{SD}}^{-10}gy^{10}\frac{GM}{r^2}=0\,, \tag{17}$$

where $y_{\mathrm{SD}}=(z_0/r)_{\mathrm{SD}}$. The structure of the accretion disk is to be calculated from Eqs. (15), (16) and (17) with appropriate boundary conditions at the outer edge of the disk.

In order to demonstrate that SADM approximation is not valid even for geometrically thin accretion disk, let us consider a sufficiently thin accretion disk. In such a disk without changing essential nature of basic equations, we can neglect the $y^2$ and $y^4$ terms in the brackets of Eq. (15) and the first two terms in Eq. (17). The $y^4$ term in Eq. (15) is to shift slightly the zero-point of the derivative. The $y^2$ term in the demoninator plays an important contribution only if $\alpha^2Ay^2g^2\gtrsim1$, that is, if $g\propto y^{-1}\gg1$. Since we are interested in $f$ and $g$ in ranges not much different from unity, the neglect of these terms gives rise to no serious error. It is noted that the above approximations are equivalent to neglect the $v_r$ term in Eqs. (2) and (4) and the first term in Eq. (4), i.e., the term referred to the internal energy plus work done by pressure. Consequently, from Eq. (17) $y$ can be solved as

$$y=y_{\mathrm{SD}}\left\{\frac{2}{3}\,\frac{1-f^2}{g}+f^2\right\}^{1/10}\,. \tag{18}$$

In relevant ranges of $\dot{M}$ and $\alpha$, $y$ is of order $10^{-1}\sim10^{-2}$, that confirms the thin disk approximation. Furthermore, Eq. (18) indicates that $y$ is a slowly varying function of $r$, $f$ and $g$, so that $y$ can practically be put to a constant, since such an approximation does not alter the general property of Eqs. (15) and (16). Equations (15) and (16) are combined to give

$$\frac{d\ln g}{d\ln f}=\frac{2}{3A}\frac{1}{y^2}\frac{1-f^2}{1-g}-\frac{1}{1-g}\,. \tag{19}$$

By virtue of $y=$const., Eq. (19) is solved as

$$\ln g-g+\frac{1}{3A}\frac{1}{y^2}f^2-\left(\frac{2}{3A}\frac{1}{y^2}-1\right)\ln f=C\,, \tag{20}$$

where $C$ is the integration constant.

Solutions to Eq. (19) are characterized by $C$ as shown in Fig. 1. In Fig. 1 we tentatively assume $y=10^{-1}$ and $N=3$ for the polytropic index. As seen from Fig. 1 solutions to Eq. (19) are classified into two families whether the constant $C$ is larger than the critical value given by

$$C_{\rm cr}=\frac{1}{3A}\frac{1}{y^2}-\frac{3}{2}-\frac{1}{2}\left(\frac{2}{3A}\frac{1}{y^2}-1\right)\ln\left(1-\frac{3}{2}Ay^2\right), \tag{21}$$

or not. It is to be noticed that none of these solutions converges toward $g=1$ and $f=1$ (toward SADM approximation) or approaches in the vicinity of $g=f=1$ except for one special solution, $g=1$ and $f=\sqrt{1-3Ay^2/2}$. This is nothing but the solution to SADM taking into account the $y^2$ term (see Eq. (12)). However, if bound-

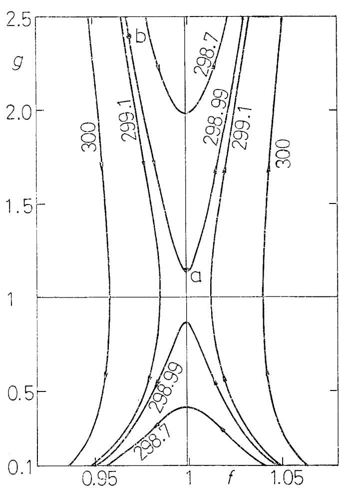


Fig. 1.

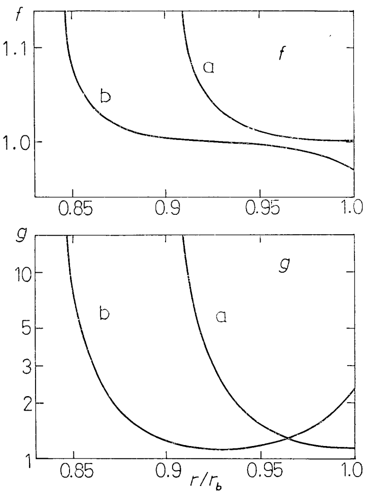


Fig. 2.

Fig. 1. Plots of solutions to Eq. (19) for the case $y=10^{-1}$ and $N=3$. Numeric attached in each curve is the value of $C$. Solutions are classified into two families depending on the value of $C$ with a critical value $C_{\rm cr}=298.9996$. When solved inward, $f$ and $g$ vary along the direction indicated.

Fig. 2. Changes of $g$ and $f$ as a function of radial distance $r$. These two curves belong the same value of $C$ but start with different boundary conditions at the outer edge of the disk. The starting points of these two solutions are indicated in Fig. 1 by (a) and (b) on the curve with $C=298.99$. In the Figure the radial distance is normalized by the radius of the outer edge, $r_b$. The gradient of $g$ becomes increasingly large at $r=0.845r_b$ and $0.905r_b$. Boundary values of $f$ and $g$ at $r=r_b$ are; $f_b=0.99$, $g_b=1.145$ for the curve a, $f_b=0.97$, $g_b=2.401$ for the curve b.

ary values of $g$ and $f$ at the outer edge of the disk differ infinitesimally from the above values, the solution never converges to SADM.

In Fig. 2 we plot two different solutions of $f$ and $g$ as functions of $r$ for the case of the same $C(=298.99)$ but different boundary values $f$ and $g$ at the outer edge of the disk. Since Eq. (15) is proportional to $y^{-2}$, the variation of $g$ is extremely large as compared with that of $f$. This means that the azimuthal velocity is nearly Keplerian but the pressure differs largely from that of SADM. Extreme growth of $g$ prevents further inward numerical integration beyond $r=0.845\,r_b$ and $0.905\,r_b$ (where $r_b$ denotes the outer edge of the disk).

## § 5. Concluding remarks

In the preceding section we have found that none of solutions fulfills SADM approximation except for the special solution, $g=1$ and $f=\sqrt{1-3Ay^2/2}$. In deriving the above conclusion we have used several approximations. However, in the ranges of $f$ and $g$ illustrated in Fig. 1, we can easily verify that original equations (15), (16) and (17), provide no new result other than shown in § 4.

According to Shibazaki, preliminary numerical calculations indicate that optically thick accretion disks dominated by radiation pressure also provide the same property as that of the present analysis. One of main causes lies in the viscous stress adopted in the present paper, which is proportional to the pressure $W$ as in Eq. (7). In terms of Eqs. (5) and (7), Eqs. (2) and (3) can be written as

$$\frac{d\ln W}{d\ln r}=\frac{1}{Ay^2}\left(v_\varphi{}^2/\frac{GM}{r}-1\right), \tag{22}$$

$$\frac{d\ln v_\varphi}{d\ln r}=-\frac{3\dot{M}}{4\pi\alpha}\left(\frac{GM}{r}\right)^{1/2}\frac{1}{W}+1\,, \tag{23}$$

where we have neglected the first term of Eq. (2). Consider the case that $v_\varphi{}^2$ is slightly larger than $GM/r$, then, according to Eq. (22) a considerable amount of pressure gradient is required, since $y^{-2}\gg 1$, which gives rise to decrease of pressure at an adjacent inner zone of the disk. From Eq. (23) the decrement of pressure results in a larger $v_\varphi$ at that zone. A larger $v_\varphi$ results in a further smaller pressure at an inner zone. If the viscous stress were proportional to, for example, $W^{-1}$, contrary to the case discussed above, solutions converge toward $g=f=1$.

In the present paper, we have failed to solve a complete structure of a stationary accretion disk extending into the vicinity of the black hole. It is uncertain, however, whether stationary accretion disks are theoretically possible or not, since only limited attempts have been made to find solutions to basic equations discussed. If exists, its structure may distinctly be different from that of SADM.

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